# Studies on Differential Equations 

by<br>Dr. M. Syed Ali

Assistant Professor
Department of Mathematics
Thiruvalluvar University
Vellore
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## Introduction

- Theoritical formulation of a physical system
- These formulations turn out to be "Differential Equations"
- D.E - bridge Mathematics and Science with its applications
- Differential Equations = Derivatives connected with Equations
- Unknowns are connected through the concept of derivatives.
- ODE - Involving ordinary derivatives of an unknown function


## Ordinary Differential Equation

## Definition

ODE: An ordinary differential equation of order n is defined by the relation

$$
F\left(x, y, y^{1}, y^{2}, \ldots y^{n}\right)=0,-----------(1)
$$

where $y(n)$ stands for nth derivative of unknown function $x \rightarrow y(x)$ with respect to the independent variable $x$.

## Definition

Initial Value Problem for an ODE: Let $x_{0} \in I$ and $\left(y_{1}, y_{2}, \ldots, y_{n}\right) \in \Omega$ be given. An Initial Value Problem (IVP) for an ODE in normal form is a relation satisfied by an unknown function y given by

$$
y^{n}=f\left(x, y, y^{1}, y^{2}, \ldots y^{n-1}\right), y^{i}\left(x_{0}\right)=y_{i}, i=0, \ldots,(n-1)
$$

## Solution

Having defined an ODE, we are interested in its solutions. In algebra, solution of an algebraic equation

$$
a x^{2}+b x+c=0, a \neq 0
$$

The real or complex number is a solution if x satisfies above.
In D.E solution is a function
$x(t)$ is solution if
$x(t)$ to be differentiable
$x(t)$ should satisfy the concerned equation

## Definition

Solution of an ODE: A real valued function $\phi$ is said to be a solution of ODE (1) if $\phi \in C^{n}(I)$ and

$$
F\left(x, \phi(x), \phi^{1}(x), \phi^{2}(x), \ldots \phi^{n}(x)\right)=0, \quad \forall x \in I
$$

## Solving Methods

First order Equations

- method of variation of parameters
- Seperable equations
- Exact Equations
$\mathbf{N}^{\text {th }}$ orderEquations
The general $n^{\text {th }}$ order equation is described by

$$
x^{n}+b_{1} x^{n-1}+\ldots+b_{n} x=0, \quad b_{1}, b_{2}, \ldots \text { are real or complex }
$$

Let $x(t)=e^{\lambda t}$ is solution

$$
\left(e^{\lambda t}\right)\left[\lambda^{n}+b_{1} \lambda^{n-1}+\ldots+b_{n}\right]=0
$$

Since $e^{\lambda t} \neq 0$ then

$$
P(\lambda)=\lambda^{n}+b_{1} \lambda^{n-1}+\ldots+b^{n}=0
$$

Hence $e^{\lambda t}$ is solution If $\lambda$ is a root of $P(\lambda)=0$ then.
The roots of Eqn $P(\lambda)=0$ are charecteristic roots.

There are three possibilities

- $\lambda$ are real and distinct
- $\lambda$ are real and equal
- $\lambda$ are complex


## Theorem

If $\lambda$ is a root of equation $P(\lambda)=0$ then $e^{\lambda t}$ is solution of ( ${ }^{*}$ ). In $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ are distinct we get $n$ solutions

$$
\phi_{1}(t)=e^{\lambda_{1} t}, \ldots, \phi_{n}(t)=e^{\lambda_{n} t}
$$

Their linear combination

$$
c_{1} \phi_{1}+\ldots+c_{n} \phi_{n}
$$

is a solution.

## Theorem

Suppose $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ are repeated $m_{1}, m_{2}, \ldots m_{s}$ times ( wherem $_{1}+m_{2}+\ldots+m_{s}=n$ then $n$ functions

$$
\phi_{1}=e^{\lambda_{1} t}, t e^{\lambda_{1} t}, \ldots, t_{1}^{m-1} e^{\lambda_{1} t}\left(m_{1} \text { functions }\right)
$$

$$
\phi_{n}=e^{\lambda_{s} t}, t e^{\lambda_{s} t}, \ldots, t_{s}^{m-1} e^{\lambda_{s} t}\left(m_{1} \text { functions }\right)
$$

are solutions. Their linear combination

$$
c_{1} \phi_{1}+\ldots+c_{n} \phi_{n}
$$

is a solution.

## Theorem

Suppose $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ are complex roots $\lambda_{j}=\alpha+i \beta$ is complex root with multiplicity $m_{j}$ then $\lambda_{j}=\alpha-i \beta$ is also a complex root with mulitiplicity $m_{j}$ and correspondng 2 mj solutions are given by

$$
\begin{aligned}
& e^{\alpha t} \cos \beta t, t e^{\alpha t} \cos \beta t, \ldots, t_{s}^{m_{j}-1} e^{\alpha t} \cos \beta t \\
& e^{\alpha t} \sin \beta t, t e^{\alpha t} \sin \beta t, \ldots, t_{s}^{m_{j}-1} e^{\alpha t} \sin \beta t
\end{aligned}
$$

Some Standard Methods are

- Method of undetermined co efficients
- Reduction of the order of Equation
- Method of Laplace Transforms


## Systems of ODE

## Definition

System of ODEs: A first order system of n ordinary differential equations is given by

$$
y^{\prime}=f(x, y)-------(2)
$$

## Definition

Linear Autonomous systems The linear simplest systems are described by

$$
\dot{x}=A x, x(0)=\bar{x}
$$

where $x:[0, T] \rightarrow R^{n}$ is assumed to be continuously differrentiable. $A \in L\left(R^{n}\right)$ and is represented by a matrix.

## Example

Consider the damped harmonic oscillator

$$
\ddot{y}+k \dot{y}+\omega^{2} y=0, y(0)=x_{0}, \dot{y}=x_{1}
$$

introducing $y=x_{1}, \dot{y}=x_{2}$ we find

$$
\begin{gathered}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-k x_{2}^{2}-\omega^{2} x_{1}
\end{gathered}
$$

or

$$
\dot{x}=\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-\omega^{2} & -k
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]
$$

## Nonlinear Ordinary Differential Equations

Consider the system

$$
\dot{x}=f(t, x) \quad x(\tau)=\bar{x}
$$

wherex is a vector valued function $x: I=[\tau, \tau+a] \rightarrow R^{n}$ and $f: D=I \times B \rightarrow R^{n}$ where

$$
B=\left\{x \in R^{n}: \| x-\bar{x} \leq b\right\}
$$

Solution We define $x(t)$ to be a solution if

- $(t, x(t)) \in D$ for $t \in I$
- $x(t)$ satisfies the concerned equation for all $t \in I$ that is

$$
\dot{x}(t)=f(t, x(t)), \quad x(\tau)=\bar{x}
$$

Let us assume $f$ is continuous on $d$, then $x$ to be a solution it must be continuously differentiable for all $t \in I$ and must satisfy

$$
x(t)=\bar{x}+\int_{\tau}^{t} f(s, x(s)) d s
$$

## Basic questions

There are three basic questions associated to problems. They are

- Does the problem admit at least one solution?
- Assuming that the problem has a solution, is the solution unique?
- Assuming that the problem admits a unique solution y on a common interval I, what is the nature or behaviour of the solution?


## Existence theorems

In general we are interested in the following questions:

- Under what conditions can we be sure that a solution exists to the problem?
- Under what conditions can we be sure that there is a unique solution? Here are the answers.


## Theorem

Let $f: I \times R^{n} \rightarrow R^{n}$ be continuous. Assume that there exist two continuous functions $h, k: I \times R_{+}$(non-negative real-valued) such that

$$
\|f(x, y)\| \leq k(z)\|y\|+h(x), \forall(x, y) \in I \times R^{n}
$$

Then for every initial data $\left(x_{0}, y_{0}\right) \in I \times R^{n}$, IVP has at least one global solution.

## Theorem

Existence: Suppose that $F(x ; y)$ is a continuous function defned in some region $R=\left\{(x, y): x_{0}-\delta<x<x_{0}+\delta, y_{o}-\epsilon<y<y_{0}+\epsilon\right\}$ containing the point $\left(x_{0}, y_{0}\right)$. Then there exists a number $\delta_{1}$ (possibly smaller than $\delta$ ) so that a solution $y=f(x)$ exists to $y^{\prime}=f(x, y)$ is defined for $x_{0}-\delta_{1}<x<x_{0}+\delta_{1}$.

## Theorem

Uniqueness: Suppose that both $F(x, y)$ and $\frac{\partial F}{\partial y}(x, y)$ are continuous functions defined on a region $R$ as in Theorem. Then there exists a number $\delta_{2}$ (possibly smaller than $\delta_{1}$ ) so that a solution $y=f(x)$ exists to $y^{\prime}=f(x, y)$ whose existence was guaranteed by Theorem 1, is the unique solution for $x_{0}-\delta_{2}<x<x_{0}+\delta_{2}$.

## Example

Consider the ODE

$$
y^{\prime}=x-y+1 ; \quad y(1)=2
$$

In this case, both the function $F(x ; y)=x-y+1$ and its partial derivative $\frac{\partial F}{\partial y}(x, y)=-1$ are defined and continuous at all points $(x ; y)$. The theorem guarantees that a solution to the ODE exists in some open interval centered at 1 , and that this solution is unique in some (possibly smaller) interval centered at 1.

## Brouwer Theorem

In the plane

## Theorem

Every continuous function from a closed disk to itself has at least one fixed point.

This can be generalized to an arbitrary finite dimension: In Euclidean space

## Theorem

Every continuous function from a closed ball of a Euclidean space to itself has a fixed point.

A slightly more general version is as follows Convex compact set

## Theorem

Every continuous function from a convex compact subset $K$ of a Euclidean space to K itself has a fixed point.

## Banach Contraction Principle

## Definition

(Contraction). Let $(X ;\|\cdot\|)$ be a normed linear space. A function $F: X \rightarrow X$ is called a contraction if there exists a $\mathrm{k}, \mathrm{k}<1$ such that

$$
\|F(x)-F(y)\| \leq k\|x-y\|
$$

for all $x, y \in X$

## Theorem

Let $X$ be a Banach space and $F: X \rightarrow X$ a contraction. Then there is a unique $x^{*}$ such that

$$
F\left(x^{*}\right)=x^{*}
$$

$x^{*}$ is called the fixed point of $F$.

## Schauder theorem

## Theorem

Schauder fixed point theorem If $K$ is a convex subset of a topological vector space $V$ and $T$ is a continuous mapping of $K$ into itself so that $T(K)$ is contained in a compact subset of $K$, then $T$ has a fixed point.

## Theorem

Leray - Schauder Let $D$ a convex subset of a Banach space $X$ and assume that $0 \in D$. Let $F: D \rightarrow D$ be a completely continious map. then the set

$$
\{x \in D: x=\lambda F(x), 0<\lambda<1\}
$$

is unbounded or the map $F$ has a fixed point in $D$.

## Schafer Theorem

## Theorem

Let $E$ be a normed linear space. Let $F: E \rightarrow E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set contained in a compact set, and let

$$
F=\{x \in E: x=a F x, 0<x<1\}
$$

Then either $F$ is unbounded or the map F has a fixed point.

## Krasnoselskii Theorem

## Theorem

Let $M$ be a closed convex nonempty subset of a Banach space $(B,\|\cdot\|)$. Suppose that $A$ and $B$ map $M$ into $B$ such that

- (i) $x, y \in M \Rightarrow A x+B y \in M$,
- (ii) $A$ is compact and continuous,
- (iii) $B$ is a contraction mapping.

Then there exists $y \in M$ with $y=A y+B y$.
When A is the zero operator, this is Banach's fixed point theorem; when B is zero, this is Schauder's fixed point theorem.

## Arzela-Ascoli Theorem

## Theorem

If $X$ is compact and $F \subseteq C(X)$ then $F$ is totally bounded iff $F$ is bounded and equicontinuous

## Definition

If $X$ is completely regular and $F \subseteq C(X)$ then F is equicontinuous if for every $\epsilon>0$ and for every $x_{0}$ in $X$ there is a neighbourhood $U$ of $x_{0}$ such that

$$
\left|f(x)-f\left(x_{0}\right)\right|<\epsilon
$$

for all $x$ in $U$ and for all $f$ in $F$

## Numerical Solutions

## Some Standard Methods for Numerical Solutions are

- Euler method,
- Runge-Kutta method
- Runge Kutta Butcher Method
- Predictor - Corrector method
- Asams - Bashforth Method


## Euler Method

## Consider the problem

$$
x,=f(x, t)
$$

The algorithm is
Let $t_{i}=a+i h$

$$
X_{k+1}=X_{k}+h f_{k}, k=0,1,2, \ldots, N-1
$$

Where $X_{0}=x_{0}$ and $f_{k}=f\left(t_{k}, x_{k}\right), h=(b-a) / N$

## Runge - Kutta Method

Consider the problem

$$
x,=f(t, x), x(a)=x_{0}
$$

The algorithm is
Let $t_{n}=a+n h, n=0,1,2, \ldots, N-1$

$$
x_{n+1}=x_{n}+\frac{1}{6}\left(k_{1}+2 k_{2}+2 k_{3}+k_{4}\right)
$$

Where $X_{0}=x_{0}$ and $f_{k}=f\left(t_{k}, x_{k}\right), h=(b-a) / N$ $k_{1}=h f\left(t_{n}, X\right)$ $k_{2}=h f\left(t_{n}+\frac{h}{2}, X+\frac{k_{1}}{2}\right)$
$k_{3}=h f\left(t_{n}+\frac{h}{2}, X+\frac{k_{2}}{2}\right)$
$k_{4}=h f\left(t_{n}+h, X+k_{3}\right)$

## Stochastic Differential Equaitons

Very often there are some random fluctuations or noise associated with Physical systems, and in order to represent we are led to stochastic models.
The more realistic model for linear system wuld be

$$
\dot{x}=A x+\xi
$$

where $\xi$ is some noise process, often called "'white noise"'.
Consider a systems where noise accur additively and is of the form

$$
\dot{x}=A(t) x(t)+B(t) \dot{w}, x(\tau)=\bar{x}, t \in[\tau, T]
$$

where $x(t) \in R^{n}$ for each $t, A(t)$ is a $n \times n$ matrix, mesearuble and bounded on $[\tau, T]$ and $\mathrm{B}(\mathrm{t})$ is an $n \times n$ matrix measurable and bounded on $[\tau, T]$. $\dot{w}(t)$ is a symbolic represntation of Wiener process.
The integral equation is of the form

$$
x(t)=\bar{x}+\int_{\tau}^{t} A(s) x(s) d s+\int_{\tau}^{t} B(s) x(s) d w(s)
$$

## Example

Consider the simple population growth model

$$
\begin{equation*}
\frac{d X(t)}{d t}=a(t) X(t) \tag{1}
\end{equation*}
$$

with initial value $X(0)=X_{0}$, where $X(t)$ is the size of the population at time $t$ and $a(t)$ is the relative rate of growth. It might happen that $a(t)$ is not completely known, but subject to some random environmental effects. In other words,

$$
a(t)=r(t)+\sigma(t) \text { "noise", }
$$

so equation (1) becomes

$$
\frac{d X(t)}{d t}=r(t) X(t)+\sigma(t) X(t) \text { "noise". }
$$

That is, in form of integration,

$$
X(t)=X_{0}+\int_{0}^{t} r(s) X(s) d s+\int_{0}^{t} \sigma(s) X(s) \text { "noise"ds. }
$$

## Delay Differential Equations

Systems depends on the past state.
In economics, biology, control ettc past influences the future.
For example, if we model a population by assuming that the growth is proportional to the number in the population between the ages of 15 and 45 , then a possible model is

$$
N(t)=k[N(t-15)-N(t-45)]
$$

Where $N(t)$ is the number in the population at time t , Then it is useful to define the function $N_{t}$ by

$$
N_{t}(\theta)=N(t+\theta), \quad-45 \leq \theta \leq 0
$$

Genereally, we define $x_{t}(\theta)=x(t+\theta),-r \leq \theta \leq 0$
The general delay diffrential equation is of the form

$$
\dot{x}(t)=f(t, x(t), x(t-\tau))
$$

## Thank You

